Non-Linearity of Portfolio Optimization

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Abstract
This note is an in-depth study of constrained mean-variance optimization in the context of combining several systematic trading signals. We analyze whether the solution of such optimization depends linearly on the input variables. The conclusion is the contrary that such portfolio optimization exhibits a multitude of non-linearity. We conclude by discussing implications for investors.

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Portfolio Optimization

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1. Introduction

The goal of portfolio optimization is to determine an optimal combination of assets within a portfolio according to some objective. An optimal portfolio can have many advantages, for example, lower volatility, higher risk-adjusted returns or balanced exposures. A critical development was *modern portfolio theory*, first described by Markowitz (1952), which introduced the idea of mean-variance analysis and a mathematical framework for assembling a portfolio of assets such that the expected return is maximized for a given level of risk. In the years since, mean-variance optimization (MVO) has become a commonly used portfolio construction method. However, optimizing a portfolio can come at the cost of transparency – particularly in terms of portfolio attribution. In this Research Note, we provide some mathematical results for why it is not always possible to precisely attribute performance in optimal portfolios.

Three Key Equations

In a typical systematic trading strategy, several signals are supplied as expected returns $\mathbf{r}_i$, which are combined linearly into a single set $\bar{\mathbf{r}} = \sum_i a_i \mathbf{r}_i$ using signal weights a_i. The portfolio is then optimized subject to a volatility target, plus one or more linear constraints such as total capital usage, maximum position size limits, etc. Denote by $\mathbf{w} = \mathcal{O}(\mathbf{r})$ the function that associates an optimized portfolio $\mathbf{w}$ with a set of expected returns $\mathbf{r}$. One might reasonably expect several forms of linear behavior:

$$\mathcal{O}(\mathbf{r}) \propto \mathbf{M}\mathbf{r}, \text{ for some matrix } \mathbf{M}, \tag{1}$$

which intuitively means that a stronger/weaker expected return leads to a proportionally bigger/smaller position;

$$\mathcal{O}\left(\sum_i a_i \mathbf{r}_i\right) = \sum_i c_i \mathcal{O}(\mathbf{r}_i), \text{ for some } c_i, \tag{2}$$

which expresses the optimized blended portfolio as a linear combination of optimized component portfolios;

$$[c_i] \propto [a_i]\,, \text{ for the } c_i, a_i \text{ above}, \tag{3}$$

which means that a signal's contribution in the optimized portfolio is proportional to its contribution in expected returns.

In the pages that follow, we work progressively to show that all of these expectations can be rejected. Non-linearity is present at multiple levels in portfolio optimization.

2. MVO with Volatility Target

Consider a set of expected returns $\mathbf{r}$ for a portfolio of assets having market covariance matrix $\mathbf{\Sigma}$.

It is well known that in the absence of constraints we can solve the MVO problem with portfolio weights $\mathbf{w}$ such that $\mathbf{w} \propto \mathbf{\Sigma}^{-1}\mathbf{r}$. Note that we can set $\mathbf{M} = \mathbf{\Sigma}^{-1}$ to satisfy the linearity condition in Equation 1, which provides a motivation for the results that follow.

More specifically, if we want a portfolio with volatility σ we can specify the problem as:

$$\max_{\mathbf{w}} \mathbf{r}'\mathbf{w}, \text{ such that } \mathbf{w}'\mathbf{\Sigma}\mathbf{w} \le \sigma^2, \tag{4}$$

for which there exists a closed-form solution[1] for the allocation weights:

$$\mathbf{w} = \frac{\sigma}{\sqrt{\mathbf{r}'\mathbf{\Sigma}^{-1}\mathbf{r}}}\mathbf{\Sigma}^{-1}\mathbf{r}. \tag{5}$$

Now suppose we have two sets of expected returns $\mathbf{r}_1$ and $\mathbf{r}_2$. For each set we can solve the MVO problem:

$$\mathbf{w}_1 = \frac{\sigma}{\sqrt{\mathbf{r}_1'\mathbf{\Sigma}^{-1}\mathbf{r}_1}}\mathbf{\Sigma}^{-1}\mathbf{r}_1,$$
$$\mathbf{w}_2 = \frac{\sigma}{\sqrt{\mathbf{r}_2'\mathbf{\Sigma}^{-1}\mathbf{r}_2}}\mathbf{\Sigma}^{-1}\mathbf{r}_2.$$

If we form a blended portfolio as a combination of $\mathbf{r}_1$ and $\mathbf{r}_2$ denoted by $\bar{\mathbf{r}} = a_1\mathbf{r}_1 + a_2\mathbf{r}_2$, we can similarly solve for the MVO weights as:

$$\bar{\mathbf{w}} = \frac{\sigma}{\sqrt{\bar{\mathbf{r}}'\mathbf{\Sigma}^{-1}\bar{\mathbf{r}}}}\mathbf{\Sigma}^{-1}\bar{\mathbf{r}}$$
$$= \frac{\sigma\mathbf{\Sigma}^{-1}(a_1\mathbf{r}_1 + a_2\mathbf{r}_2)}{\sqrt{(a_1\mathbf{r}_1 + a_2\mathbf{r}_2)'\mathbf{\Sigma}^{-1}(a_1\mathbf{r}_1 + a_2\mathbf{r}_2)}}.$$

[1] See Appendix A.

From which it follows immediately that $\bar{\mathbf{w}} \neq a_1\mathbf{w}_1 + a_2\mathbf{w}_2$, rather:

$$\bar{\mathbf{w}} = \frac{L_1}{\bar{L}} a_1 \mathbf{w}_1 + \frac{L_1}{\bar{L}} a_2 \mathbf{w}_2,$$

where we define

$$L = \sqrt{\mathbf{r}'\mathbf{\Sigma}^{-1}\mathbf{r}},$$

and

$$\bar{L} = \sqrt{\bar{\mathbf{r}}'\mathbf{\Sigma}^{-1}\bar{\mathbf{r}}} = \sqrt{(a_1\mathbf{r}_1 + a_2\mathbf{r}_2)'\mathbf{\Sigma}^{-1}(a_1\mathbf{r}_1 + a_2\mathbf{r}_2)}.$$

The post-MVO weights between the two sets change from a_i to $c_i = (L_i/\bar{L})a_i$. Note that as $\bar{L}$ depends on a_i, c_i is not a linear transformation of a_i, which means that MVO with volatility target does not satisfy the linearity condition in Equation 3.

3. Adding a Linear Constraint

Consider adding a linear constraint to our MVO problem with volatility target:

$$\max_{\mathbf{w}} \mathbf{r}'\mathbf{w}, \text{ such that } \begin{cases} \mathbf{w}'\mathbf{\Sigma}\mathbf{w} \leq \sigma^2, \\ \mathbf{k}'\mathbf{w} \leq b. \end{cases} \tag{6}$$

For example we can set up a maximal capital usage constraint by setting $\mathbf{k} = \mathbf{1}$, the unit vector.

For ease of notation, it is convenient to define several auxiliary variables:

$$F = \mathbf{k}'\mathbf{\Sigma}^{-1}\mathbf{k},\ G = \mathbf{r}'\mathbf{\Sigma}^{-1}\mathbf{r},\ H = \mathbf{k}'\mathbf{\Sigma}^{-1}\mathbf{r}, \tag{7}$$

$$J = \sqrt{\frac{\sigma^2 F - b^2}{FG - H^2}}, \text{ if } \sigma^2 F - b^2 \geq 0. \tag{8}$$

When the volatility target is not too tight relative to the linear constraint (expressed by the technical condition $\sigma^2 F - b^2 \geq 0$) there exists a closed-form solution[2] for the allocation weights, which are bounded by both volatility and linear constraints:

$$\mathbf{w} = J\mathbf{\Sigma}^{-1}\mathbf{r} + \frac{b - HJ}{F}\mathbf{\Sigma}^{-1}\mathbf{k}. \tag{9}$$

As in the previous section, suppose we blend two sets of expected returns $\bar{\mathbf{r}} = a_1\mathbf{r}_1 + a_2\mathbf{r}_2$, we can form the optimized blended portfolio:

$$\begin{aligned} \bar{\mathbf{w}} &= \bar{J}\mathbf{\Sigma}^{-1}\bar{\mathbf{r}} + \frac{b - \bar{H}\bar{J}}{\bar{F}}\mathbf{\Sigma}^{-1}\mathbf{k} \\ &= \bar{J}\mathbf{\Sigma}^{-1}\left(a_1\mathbf{r}_1 + a_2\mathbf{r}_2\right) + \frac{b - \bar{H}\bar{J}}{\bar{F}}\mathbf{\Sigma}^{-1}\mathbf{k}. \end{aligned}$$

Here we see that $\bar{\mathbf{w}}$ now has a residual component proportional to $\mathbf{\Sigma}^{-1}\mathbf{k}$ that is not a linear combination of $\mathbf{r}_i$, rejecting Equation 1. We can take this analysis further by considering the individual component portfolios formed by MVO on each set of expected returns with both volatility target and linear constraint:

$$\mathbf{w}_1 = J_1\mathbf{\Sigma}^{-1}\mathbf{r}_1 + \frac{b - H_1J_1}{F_1}\mathbf{\Sigma}^{-1}\mathbf{k},$$

$$\mathbf{w}_2 = J_2\mathbf{\Sigma}^{-1}\mathbf{r}_2 + \frac{b - H_2J_2}{F_2}\mathbf{\Sigma}^{-1}\mathbf{k},$$

and ask whether there exist coefficients c_i such that

$$\bar{\mathbf{w}} = c_1\mathbf{w}_1 + c_2\mathbf{w}_2.$$

Note that all three portfolios $\bar{\mathbf{w}}, \mathbf{w}_1, \mathbf{w}_2$ are bounded by both volatility and linear constraints. The linear constraint necessarily implies that $c_1 + c_2 = 1$. However, unless $\mathbf{w}_1$ and $\mathbf{w}_2$ are equal, their correlation will be less than 1, and the volatility of $c_1\mathbf{w}_1 + c_2\mathbf{w}_2$ will be less than σ. We conclude that there must be a non-zero residual portfolio ϵ involved:

$$\bar{\mathbf{w}} = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \epsilon,$$

which means the optimal portfolio $\bar{\mathbf{w}}$ is not a linear combination of optimized component portfolios $\mathbf{w}_i$, rejecting Equation 2.

4. A Numerical Example

We present a simple example of two sets of signals for 3 assets. (Table 1). Further, we assume $\mathbf{\Sigma} = I_3$ the identity matrix, $\sigma^2 = 0.5$, $\mathbf{k} = \mathbf{1}$ and $b = 1$. We observe that the optimal solution to the portfolio of combined signals is different by a non-zero residual component ϵ compared to the combination of the individually optimized portfolios.

$a_1 = 0.5$	$a_2 = 0.5$			
$\mathbf{r}_1$	$\mathbf{r}_2$	$\bar{\mathbf{r}}$		
$\begin{bmatrix}0.6\\0.8\\0.6\end{bmatrix}$	$\begin{bmatrix}0.8\\0.6\\0.6\end{bmatrix}$	$\begin{bmatrix}0.7\\0.7\\0.6\end{bmatrix}$		
$c_1 = 0.556$	$c_2 = 0.556$			
$\mathbf{w}_1$	$\mathbf{w}_2$	$\bar{\mathbf{w}}$	$\sum_i c_i\mathbf{w}_i$	ϵ
$\begin{bmatrix}0.167\\0.667\\0.167\end{bmatrix}$	$\begin{bmatrix}0.667\\0.167\\0.167\end{bmatrix}$	$\begin{bmatrix}0.5\\0.5\\0\end{bmatrix}$	$\begin{bmatrix}0.463\\0.463\\0.185\end{bmatrix}$	$\begin{bmatrix}0.037\\0.037\\-0.185\end{bmatrix}$

Table 1. A numerical example demonstrating how a portfolio cannot be decomposed into its constiturents. Instead a residual component remains.

We can also construct a graphical representation of this result, demonstrated in Figure 1. The 3-asset portfolios that we considered can be represented by points in 3-dimensional space. The volatility target is represented by a sphere (since the equation of a sphere is given by $x^2 + y^2 + z^2 = r^2$) , which intersects with the linear constraint represented by a plane. The intersection, which is a circle, represents all possible optimized portfolios that are bounded by both constraints. The portfolios $\mathbf{w}_1$, $\mathbf{w}_2$ and $\bar{\mathbf{w}}$ are distinct points on this circle. The dotted line between $\mathbf{w}_1$ and $\mathbf{w}_2$ represent portfolios that are linear combinations $c_1\mathbf{w}_1 + c_2\mathbf{w}_2$ that lie on the linear constraint. However, it is easy to see that these portfolios are inside the sphere and fall short of the volatility target. Similarly, $\bar{\mathbf{w}}$ does not lie on the dotted line.

[2]See Appendix B.

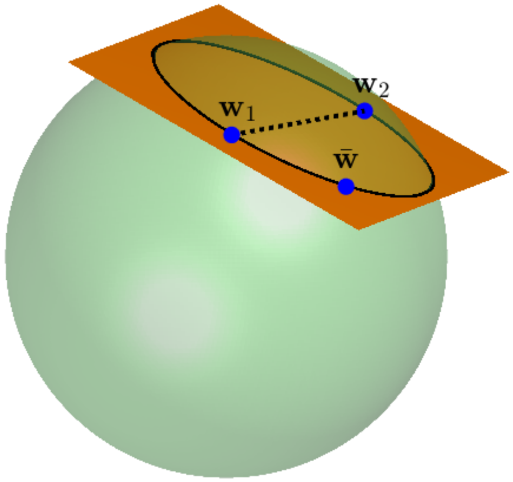

Figure 1. A grahpical representation of our 3-asset portfolio problem. Observe that no points on the line connecting $\mathbf{w}_1$ and $\mathbf{w}_2$ lie on the surface of the sphere. This means that a linear combination of $\mathbf{w}_1$ and $\mathbf{w}_2$ that satisfies the linear constraint, cannot also satisfy the volatility constraint defined by the sphere.

5. Conclusion

While portfolio optimization is a commonly used and effective tool to enhance risk-adjusted returns, its use can come with the expense of reduced transparency in terms of portfolio attribution.

In particular, when evaluating a trading strategy constructed from multiple signals combined with portfolio optimization, investors should not necessarily expect portfolio and performance attributions to add up precisely in a simple linear fashion.

Rejection of Equation 1 means *portfolio allocation weights are not directly proportional to signal strength*. Rejection of Equation 2 means *an optimized portfolio that combines several component signals has a residual piece that causes its performance to deviate from aggregated performance of underlying component strategies*. Rejection of Equation 3 means that *giving one signal a higher weighting may not lead to a commensurate increase of its contribution in the combined portfolio*.

Appendix A

Derivation of Equation 5 by solving Equation 4. For a constrained optimization problem we use Lagrangian multipliers method:

$$\mathbf{r} = 2\lambda\mathbf{\Sigma}\mathbf{w},$$

$$\mathbf{w} = \frac{1}{2\lambda}\mathbf{\Sigma}^{-1}\mathbf{r}, \tag{10}$$

$$\sigma^2 = \mathbf{w}'\mathbf{\Sigma}\mathbf{w} = \frac{1}{4\lambda^2}\mathbf{r}'\mathbf{\Sigma}^{-1}\mathbf{r},$$

$$\frac{1}{2\lambda} = \frac{\sigma}{\sqrt{\mathbf{r}'\mathbf{\Sigma}^{-1}\mathbf{r}}}, \tag{11}$$

substituting Equation 11 into Equation 10 yields the solution.

Appendix B

Derivation of Equation 9 by solving Equation 6. We use Lagrangian multipliers for two constraints:

$$\mathbf{r} = 2\lambda\mathbf{\Sigma}\mathbf{w} + \eta\mathbf{k},$$

$$\mathbf{w} = \frac{1}{2\lambda}\mathbf{\Sigma}^{-1}(\mathbf{r} - \eta\mathbf{k}), \tag{12}$$

$$\sigma^2 = \mathbf{w}'\mathbf{\Sigma}\mathbf{w} = \frac{1}{4\lambda^2}(\mathbf{r}' - \eta\mathbf{k}')\mathbf{\Sigma}^{-1}(\mathbf{r} - \eta\mathbf{k}), \tag{13}$$

$$b = \mathbf{k}'\mathbf{w} = \frac{1}{2\lambda}\mathbf{k}'\mathbf{\Sigma}^{-1}(\mathbf{r} - \eta\mathbf{k}). \tag{14}$$

Combining Equations 13 and 14 to eliminate λ, using auxiliary variables in Equation 7, we get:

$$\left(\sigma^2F^2 - b^2F\right)\eta^2 - 2\left(\sigma^2HF - b^2H\right)\eta + \left(\sigma^2H^2 - b^2G\right) = 0,$$

which is a quadratic equation of η, whose discriminant is

$$D = 4b^2(\sigma^2F - b^2)(FG - H^2).$$

Since $FG - H^2 \geq 0$ (property of $\mathbf{\Sigma}$ being a covariance matrix), $D \geq 0$ if and only if $\sigma^2F - b^2 \geq 0$, in which case the solution of η is

$$\eta = \frac{H}{F} - \frac{b}{FJ}. \tag{15}$$

Substituting this back into Equation 14 gives us

$$\lambda = \frac{1}{2J}. \tag{16}$$

Equations 12, 15, 16 together gives the closed-form solution Equation 9.

References

H. Markowitz. Portfolio selection. *The Journal of Finance*, 7(1): 77–91, 1952.

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